Mathematical Description of the Shape of Plane Hexagonal Snow Crystals

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ABSTRACT

Simple mathematical expressions are presented for describing the shapes of some plane hexagonal snow crystals. These expressions provide convenient means for cloud physical calculations and can also serve as a method for quantitative classification of snow crystal shapes. A few examples are worked out to illustrate the use of these expressions. They can be further developed for describing more complicated shapes.

1. Introduction

One of the most important physical properties of a hydrometeor is its shape. For example, the backscattering of radar waves by hydrometeors depend strongly on their shapes (Atlas *et al.*, 1953; Battan, 1973). The shape of ice crystals is of primary importance in the calculation of diffusional growth rates in the electrostatic analog theory through the capacitance factor. The shape also significantly influences the hydrodynamics of hydrometeors. Different shapes result in different flow fields and therefore different collision efficiency between hydrometeors themselves, and between hydrometeors and aerosol particles. The difference in flow fields will also result in different ventilation effects and hence influence the temperature and vapor density distributions around hydrometeors. These latter distributions are very important to the growth of ice crystals.

The shapes of hydrometeors are complicated: ranging from the spheroidal shape of a small raindrop, to the conical shape of graupel and hailstones and large raindrops; to various hexagonal plates, columns, and dendrites of ice crystals, and to highly complicated, irregular shapes of aggregates or fragments of snowflakes (Mason, 1971; Hobbs, 1974; Pruppacher and Klett, 1978). It is important for meteorologists to categorize the shapes of these particles. We address the shapes of ice crystals in this paper.

A method of classification of the shapes of natural snow crystals was given by Magono and Lee (1966). The Magono–Lee classification is a qualitative one and is useful in the descriptive categorization of snow crystals. On the other hand, in cloud physical calculations such a qualitative description is inadequate; a quantitative classification is necessary. Such a task is easy for simple shapes such as hexagonal plates and columns; one has only to specify the length (or thickness) and the radius, as has been done in literature (*e.g.*, Auer and Veal, 1970; Locatelli and Hobbs, 1974). But for other shapes it is more difficult. Recently Wang (1982) presented a simple mathematical function that can describe the shape of conical hydrometeors (conical graupel, hailstones and large raindrops). In this paper we present a mathematical method of describing the shape of hexagonal snow crystals. The same technique can also be applied to describe other (e.g., triangular) snow crystals.

2. Mathematical description

Certainly there is more than one way in describing these hexagonal shapes. However, it is believed that the method described here is one of the simplest and will also allow easy classification. Since the hexagonality represents a regular periodicity, it immediately leads us to think of the periodic sine and cosine functions. We arbitrarily choose to use sine function here. The function $\sin^2(3\theta)$ produces six peaks in the range $0 \leq \theta \leq 2\pi$. It is therefore possible to describe the hexagonal shape of snow crystals based on the function $f[\sin^2(3\theta)]$. By suitably modulating the amplitude and the width of the peak, we can generate various hexagonal snow crystal shapes. To modulate the amplitude, one has only to multiply a constant to $f \sin^2(3\theta)$. On the other hand, the width of the peak can be modulated by raising $\sin^2(3\theta)$ to a certain power b, where b can be any positive number. This is illustrated in Fig. 1. It is seen here that the peaks are broad when b is small and are narrow when b is large, because $|\sin^2(3\theta)| \leq 1$.

Based on this idea, we have arrived at two example equations in plane polar coordinates. They are described in the following:

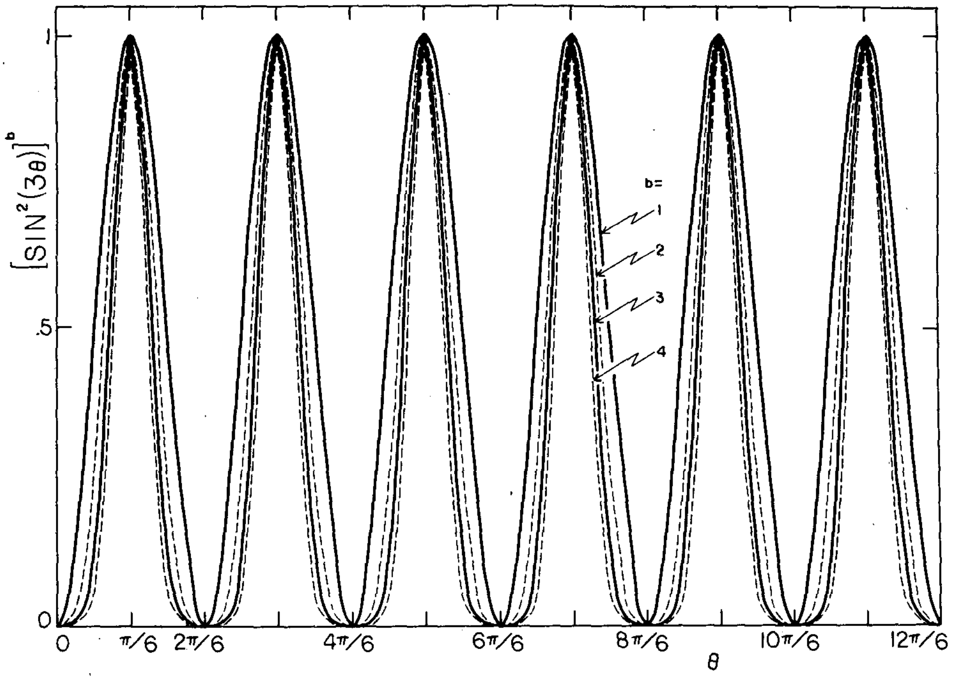

FIG. 1. Functional behavior of $(\sin^2(3\theta))^b$ versus θ.

a. Type-1

$$r = a \sin^2(3\theta)^b + C. \quad (1)$$

The constant a serves to amplify the amplitude of the peak, b to modulate the peak width. The C here represents radius of the center disk which many hexagonal snow crystals possess. Eq. (1) will generate hexagonal shapes as shown in Fig. 2. Note that the center of curvature at the tip lies inside the snow crystal which is opposite to the Type-2 case, as will be seen later.

From the geometry of Fig. 2 it is easy to see that

$$C_1 = a(1/2)^b + C, \quad (2)$$

$$C_2 = a + C, \quad (3)$$

where C_1 and C_2 are the radial lengths at $\theta = \pi/12$ and $\pi/6$, respectively. From (2) and (3) we get

$$b = \ln[(C_2 - C)/(C_1 - C)]/\ln 2. \quad (4)$$

Eqs. (3) and (4) provide all the necessary calculation in fitting Eq. (1) to a real snow crystal. The steps of fitting are:

1) Measure C, C_1 and C_2,
2) Determine a from Eq. (3),
3) Determine b from Eq. (4).

Several fitted examples are shown in Fig. 3. These are fittings of the real snow crystals in Bentley and Humphreys (1962). It is interesting to note that stellar crystals can be generated by setting a very high value to b, e.g., $b = 100$.

b. Type-2

$$r = a[1 - (\sin^2(3\theta))^b] + C. \quad (5)$$

Eq. (5) generates shapes whose centers of curvature lie outside of the snow crystals, such as the one shown

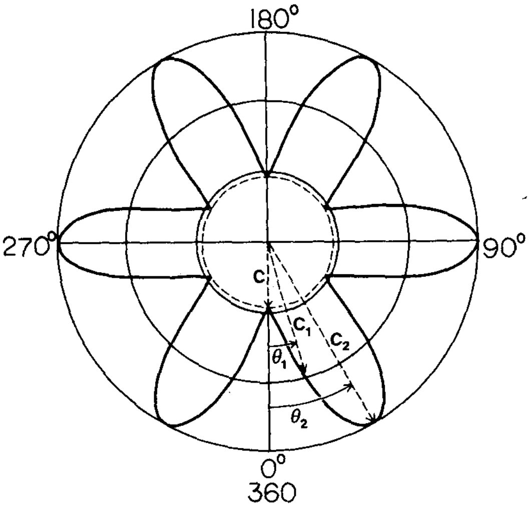

FIG. 2. Definitions of C, C_1 and C_2 for Eqs. (2) and (3). This hexagonal crystal is generated by Eq. (1) by setting $a = 5.13$, $b = 1$, $C = 2.49$ [from Bentley and Humphreys, (BH) 1962, p. 143. Crystal (3, 1)—3rd row, 1st column].

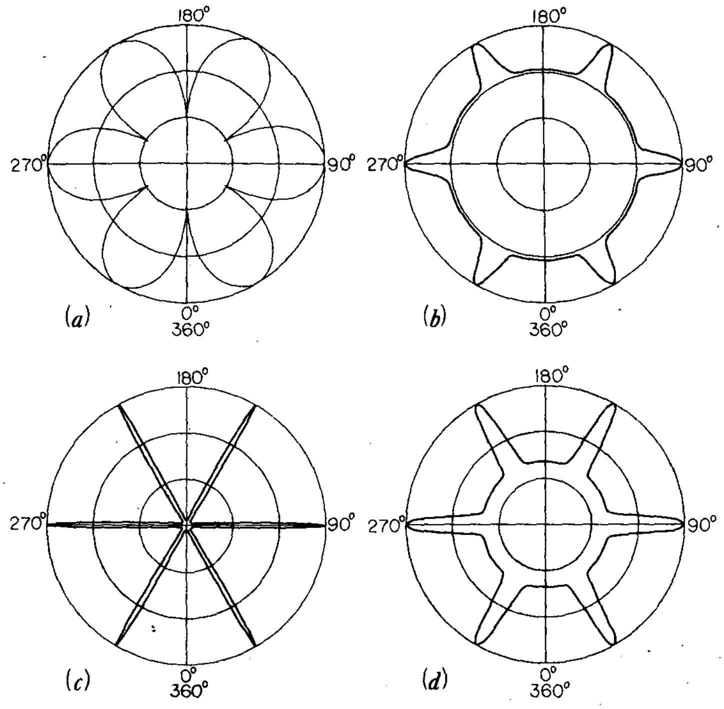

FIG. 3. Various hexagonal crystals generated by Eq. (1): (a) $a = 5.12$, $b = 0.35$, $c = 2.5$, [BH, p. 145, (4, 3)]; (b) $a = 2.45$, $b = 13$, $c = 5.17$, [BH, p. 109, (2, 3)]; (c) $a = 7.37$, $b = 50$, $c = 0.25$; (d) $a = 4.22$, $b = 12$, $c = 3.4$, [BH, p. 148, (4, 2)].

in Fig. 4. Note that the positions of the maximum dimension are rotated 90° from the previous case. Thus one of the maximum radial lengths now occurs at $\theta = 0$ while in the Type-1 case this is a minimum. From Fig. 4 we easily obtain the following relations:

$$C_0 = a + C, \tag{6}$$

$$C_1 = a(1 - 1/2^b) + C, \tag{7}$$

where C_0 and C_1 are the radial lengths of the crystal at $\theta = 0$ and $\pi/12$, respectively. C is again the radius of the center disk. From (6) and (7), we obtain

$$b = \ln[(C_0 - C)/(C_0 - C_1)]/\ln 2. \tag{8}$$

Again (7) and (8) provide the fitting formulae. The steps are

1) Measure C, C_0, C_1,
2) Determine a from Eq. (6),
3) Determine b from Eq. (8).

Several examples of this type are shown in Fig. 5. In general, Eq. (5) generates crystals with "sharp" tips. It is interesting to note that a hexagonal plate can be approximated reasonably well by Eq. (5) as shown by Fig. 5c.

The areas enclosed by Eqs. (1) and (5) can be calculated easily by the following formula:

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta, \tag{9}$$

where the r in the integrand of Eq. (9) is to be replaced

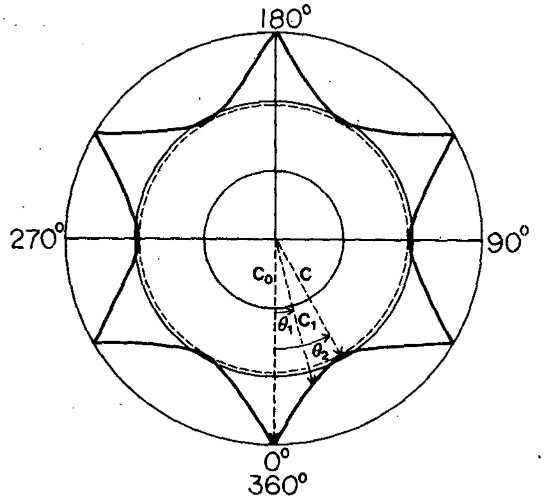

FIG. 4. Definitions of C_0, C_1 and C for Eqs. (6) and (7). This hexagonal crystal is generated by Eq. (5) by setting $a = 2.63$, $b = 0.35$, $c = 4.99$, [BH, p. 75, (2, 2)].

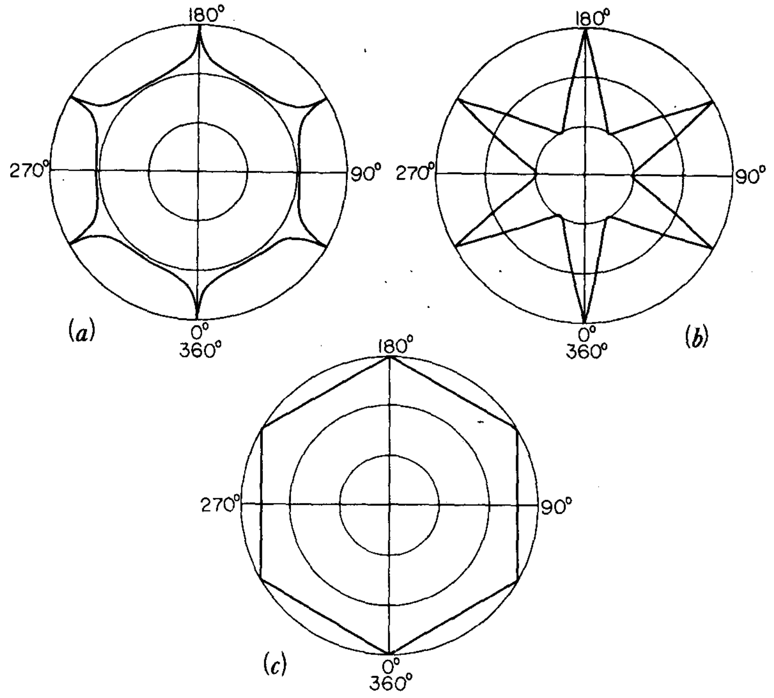

FIG. 5. Various hexagonal crystals generated by Eq. (5): (a) $a = 2.46$, $b = 0.125$, $c = 5.16$, [BH, p. 77, (1, 2)]; (b) $a = 5.12$, $b = 0.35$, $c = 2.50$, [BH, p. 148, (2, 2)]; (c) $a = 1.02$, $b = 0.35$, $c = 6.6$, [BH, p. 37].

by either Eq. (1) or (5). For a snow crystal of thickness h, the volume is simply $V = Ah$. These calculations may be easily done by numerical methods. Empirical results on the measurements of surface areas of crystals have been reported by Auer (1971). A comparison between the computation of areas using (9) and the actual measurements of Auer (1971) will be made in the future. Of course, natural ice crystals are more complicated and, therefore, when using these formulas to calculate the physical properties precautions must be taken. For example, the mass of crystals may also depend on the number of air cavities inside. The thickness of crystals may not be uniform either. The fall velocity, collection efficiency depend on many other complex factors.

3. Discussions

Eqs. (1) and (5) provide a quantitative method for classifying the observed plane hexagonal crystals. When the three parameters a, b and c are fixed, not only the size but the shape is also specified. It is felt that this method of classification is better than merely describing the dimension or the shape alone.

Although Eqs. (1) and (5) can generate many types of plane hexagonal crystals, they are by no means complete. Many other hexagonal shapes can be described in a similar manner. For example if we let

$$\gamma = a[1 - (\sin^2(3\theta))^b]^2 + C, \tag{10}$$

we obtain a shape as shown in Fig. 6, i.e., the tips are more blunt. Triangular crystals can be treated in a similar manner, provided that the periodicity is properly prescribed.

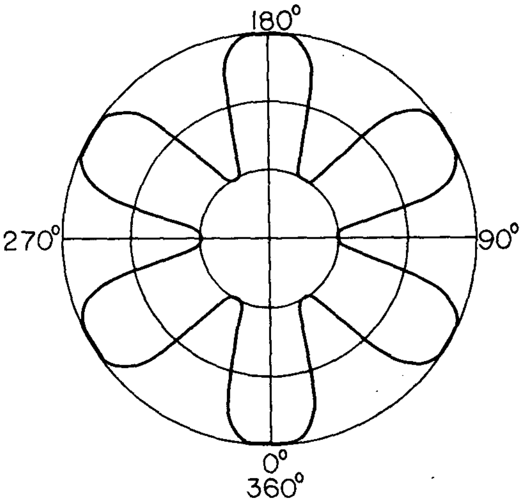

FIG. 6. Crystal generated by Eq. (10) with $a = 5.12$, $b = 2.25$, $c = 2.5$ [BH, p. 123, (3, 3)].

So far we have been dealing with simple planar hexagonal crystals with no additional branches as in the complicated dendrite case. It is possible to modify Eqs. (1), (5) and (10) to produce more complicated shapes. These will be considered in the future.

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