NOTES AND CORRESPONDENCE

Three-Dimensional Representations of Hexagonal Ice Crystals and Hail Particles of Elliptical Cross Sections

Pao K. Wang

Department of Atmospheric and Oceanic Sciences, University of Wisconsin—Madison, Madison, Wisconsin

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ABSTRACT

Mathematical expressions based on earlier ideas of the author are given to describe the three-dimensional surfaces of hexagonal ice crystals and conical graupel and hail particles. In the former class, the expressions closely parallel the expression for prolate spheroid but the cross section is transformed into a hexagon by an expression designed previously. A special transformation containing a preset constant, ε, is implemented so that the three-dimensional, instead of two-dimensional, surface of the crystal is described. In the case of conical particles with elliptical cross sections, the equation that describes a conical body of revolution is modified to describe conical bodies with elliptical cross sections.

1. Introduction

In a recent paper (Wang 1997) the author extended some of his earlier ideas (Wang and Denzer 1983; Wang 1987) to develop some formulas to represent the two-dimensional shapes of hexagonal planar ice crystals and ice columns and three-dimensional shapes of bullet rosettes and spatial dendrites. Since the representations of the hexagonal crystals (both columnar and planar) in Wang (1997) were still confined to two dimensions, they were unsatisfactory because real crystals are three-dimensional. It is now found that by combining the representations for the planar and columnar crystals we can form an expression to represent both columnar and planar ice crystals in three dimensions.

In addition, in an earlier paper (Wang 1982) the author developed an expression to represent the two-dimensional shape of conical hydrometeors (graupel, hail, and raindrops). In that paper, it was mentioned that the three-dimensional conical body of revolution could be obtained easily by rotating the 2D curve about the z axis. The horizontal cross section of such a body of revolution is a circle. Real graupel and hail particles, however, often have elliptical cross sections (Sturniolo et al. 1995; C. A. Knight and A. Waldvogel 1995, personal communication) and hence it is desirable to have formulas to represent such particles. This turned out to be an easy task; only slight modification of the previous formula is necessary. The details of the expressions for both categories of particles are given in the following sections.

Corresponding author address: Dr. Pao K. Wang, Department of Atmospheric and Oceanic Sciences, University of Wisconsin—Madison, 1225 W. Dayton Street, Madison, WI 53706-1695.
E-mail: pao@windy.meteor.wisc.edu

2. Mathematical expression for hexagonal crystals

Let us start by describing the process of obtaining the expression of a simpler shape—a right circular cylinder—and then we shall generalize that expression to represent hexagonal columns and plates.

A right circular cylinder (i.e., one with the two end surfaces perpendicular to the cylindrical side surface) can be obtained by rotating a rectangle about an axis. Wang (1997) gives the expression representing a rectangle as

$$\frac{x^2}{a^2} + \frac{z^2}{c^2}\left(1 + \varepsilon - \frac{x^2}{a^2}\right) = 1, \qquad (1)$$

where x and z are the common Cartesian coordinates; a and c are the half-lengths in the x and z direction, respectively; and ε is an adjustable positive parameter that can be set as small as we wish (but never equal to zero) to closely fit the sharp corners of a rectangle. Larger values of ε would result in ‘‘rounded’’ corners, while smaller values produce sharp corners. For regular purposes, it may be sufficient to set $\varepsilon = 10^{-5}$.

It is now easy to generate the right circular cylinder from the rectangle as represented by (1) by simply rotating it about an axis. If we now let the length be in the vertical z direction, then the cylinder is given by the following expression (see Fig. 1):

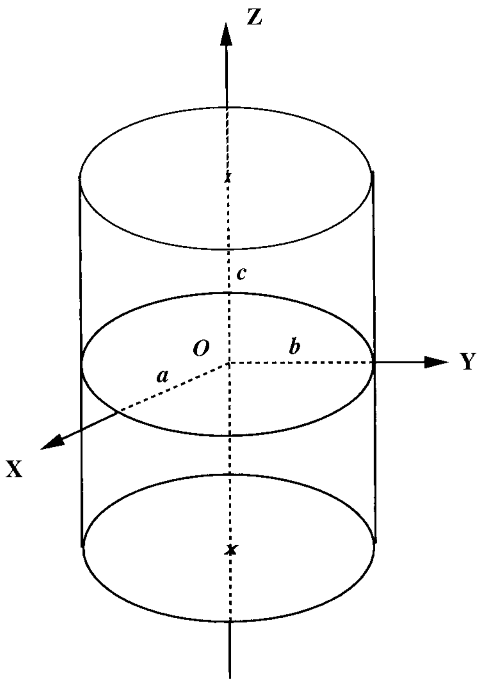

FIG. 1. A circular cylinder generated by Eq. (2).

$$\left(\frac{x^2 + y^2}{a^2}\right)\left(1 + \varepsilon - \frac{z^2}{c^2}\right) + \frac{z^2}{c^2} = 1. \tag{2}$$

This is the expression representing a right circular cylinder to a high degree of precision if ε is set to be small enough. The cross section of the cylinder in (2) is very close to a circle of radius a. The ''equator'' of the cylinder is slightly bulging but only by an amount of the order ε.

To turn this circular cylinder into a hexagonal column of length c, all we need to do is to transform the circular cross section into a hexagon. Wang (1987) has given this transformation as

$$a \to a - A[\sin^2(3\varphi)]^B, \tag{3}$$

where the expression is in 2D polar coordinates and φ is the angular coordinate. Here A and B are adjustable parameters that change the shape of the cross section. Replacing the a in Eq. (2) with the transformation in (3) would then give a column of finite length c with hexagonal cross section, except it will become an expression with mixed coordinates. Thus, it is necessary to either transform (3) into Cartesian coordinates or transform the final result into spherical coordinates. This is done in the following.

a. Cartesian coordinates representation

We note that

$$\sin(3\varphi) = 3\sin\varphi\cos^2\varphi - \sin^3\varphi$$

so that

$$\sin^2(3\varphi) = \left[\frac{3(a\sin\varphi)(a^2\cos^2\varphi)}{a^3} - \frac{a^3\sin^3\varphi}{a^3}\right]^2 = \frac{y^2(3x^2 - y^2)}{(x^2 + y^2)^3}, \tag{4}$$

where we have utilized the fact that in polar coordinates

$$\begin{cases} x = a\cos\varphi \\ y = a\sin\varphi. \end{cases} \tag{5}$$

Thus, by putting (3) into (2) but changing the resulting equation into the Cartesian form, the expression for a column of hexagonal cross section is given by

$$\frac{(x^2 + y^2)}{\left\{a - A\left[\frac{y^2(3x^2 - y^2)}{(x^2 + y^2)^3}\right]^B\right\}^2}\left(1 + \varepsilon - \frac{z^2}{c^2}\right) + \frac{z^2}{c^2} = 1. \tag{6}$$

b. Spherical coordinates representation

To express (6) in spherical coordinates, we use the conventional metrics

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta, \end{cases} \tag{7}$$

where r is the radial, θ the zenith angular, and φ the azimuth angular coordinate, respectively. By substituting the x, y, and z in (6) by (7), we get

$$\frac{r^2\sin^2\theta}{\{a - A[\sin^2(3\varphi)]^B\}^2} = \left(\frac{1 - \dfrac{r^2\cos^2\theta}{c^2}}{1 + \varepsilon - \dfrac{r^2\cos^2\theta}{c^2}}\right), \tag{8}$$

which is the desired expression.

The parameters that needed to be specified in order to generate the hexagonal cross-sectioned particle are a, c, A, and B (the parameter ε is considered to be preset). Once these four parameters are specified, both the size and shape of the particle are completely fixed. The relative lengths of a and c determine whether the particle looks more planar or columnar. If c is greater than a, then the particle is more ''columnar.'' Conversely, if a is greater than c, then it looks more ''planar.'' The parameters A and B determine the shape of the cross section, as explained in Wang (1987, 1997). Figures 2, 3, and 4 give three examples of the ice particles specified by (6) or (8). Figure 2 represents a hexagonal ice column while Fig. 3 represents a hexagonal ice plate. The only difference between the two is the length parameter c. Figure 4 represents the shape of a broad-branch crystal of the same thickness as the ice plate in Fig. 3 but of a different cross section. The readers are referred to the two earlier papers (Wang 1987, 1997) for more detailed

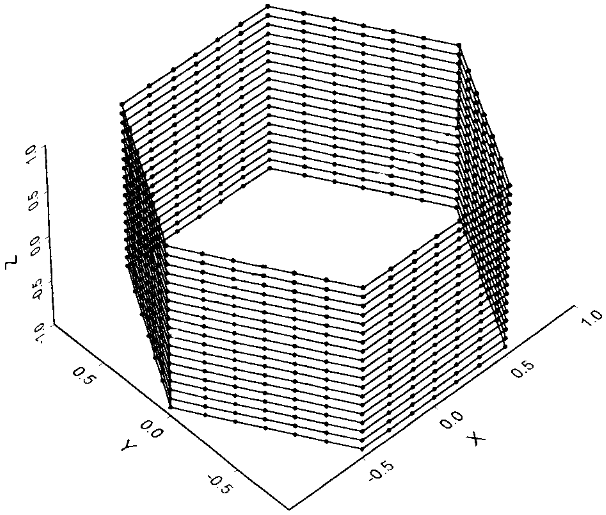

Fig. 2. A short hexagonal column generated by Eq. (6) with $a = 1$, $c = 1$, $A = 0.1339$, and $B = 0.397$. Circular dots represent the data points computed from (6). They are connected by lines to show the prism surfaces of the ice crystal. Note that (6) also generates points on the basal (top and bottom) surfaces, which are not shown here to avoid confusion.

descriptions of various cross-sectional shapes and how to generate them.

It is emphasized here that Eqs. (6) and (8) represent not only the prism surfaces but also the basal surfaces as well. The points on the basal surfaces in Figs. 2, 3, and 4 are not shown for the sake of clarity. The degree of flatness of both surfaces is controlled by the parameter ε. The smaller ε is, the closer the prism and basal surfaces to real "planes."

The surface and cross-sectional areas and volumes of the particles generated by Eqs. (6) and (8) can be easily obtained. The method of calculating the cross-sectional area of the particle is given in Wang and Denzer (1983) and Wang (1987). The volume of the ice crystal is simply the cross-sectional area times its length, c. The surface area is the sum of the basal planes (=2 × cross-sectional area) plus the area of the prism surface (=length × the perimeter of the cross section). The

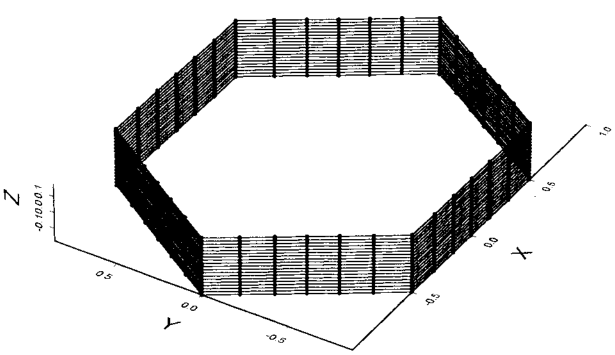

Fig. 3. A hexagonal plate generated by (6) with $a = 1$, $c = 0.2$, $A = 0.1339$, and $B = 0.397$.

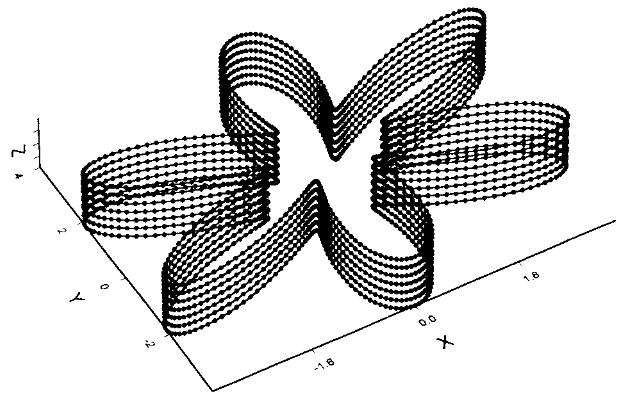

Fig. 4. A broad-branch crystal generated by (6) with $a = 1$, $c = 0.2$, $A = -3$, and $B = 1$.

perimeter length of any shape represented by (3) can be determined by the integral $\int_0^{2\pi} r\,d\varphi$, where r represents the expression in (3).

3. Conical particles with elliptical cross sections

In Wang (1982), the z-axial cross section of a conical particle is described by the following expression:

$$x = \pm a\sqrt{1 - \frac{z^2}{c^2}}\cos^{-1}\left(\frac{z}{\lambda c}\right), \qquad (9)$$

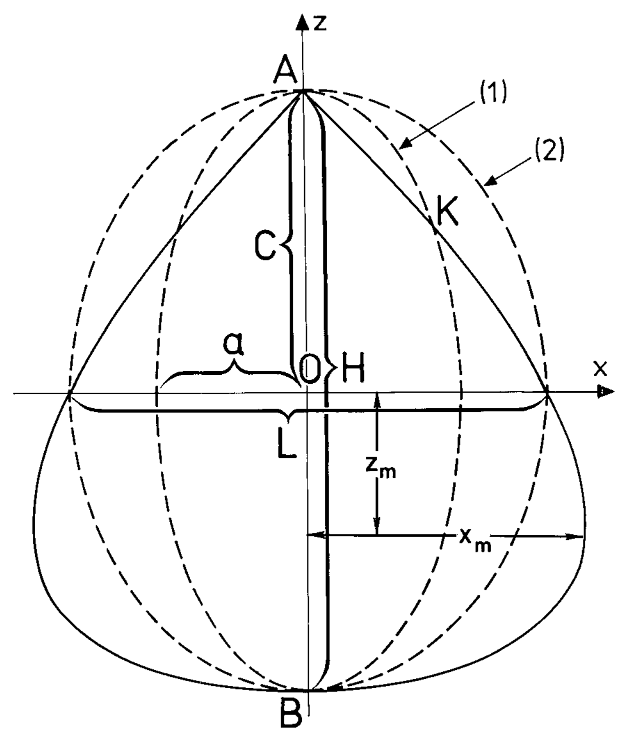

Fig. 5. Definitions of the coordinate system and various quantities appeared in Eq. (9). Solid curve is an axial cross section of a conical body. Dashed curves (1) and (2) are generating and limiting ellipses, respectively, as given in Wang (1982).

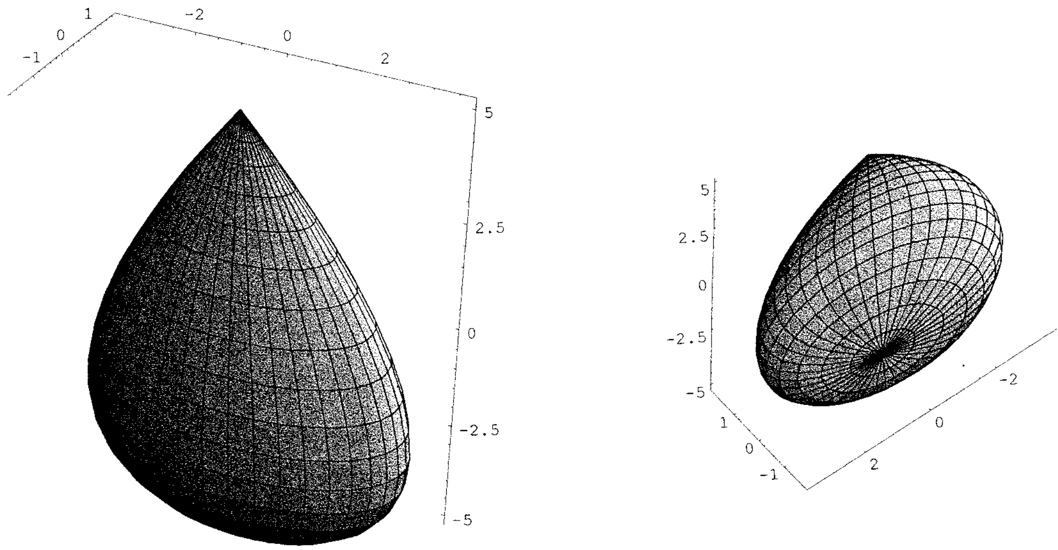

FIG. 6. A conical body with elliptical cross sections in the x–y plane as generated by Eq. (10) with $a = 2$, $b = 1$, and $c = 5$.

where the parameters a, c, and λ are defined in Fig. 5. Briefly, a and c are the semiaxes in the horizontal and vertical direction of the generating ellipse, respectively. Here λ is a parameter (with a value between 1 and ∞) that determines the ‘‘sharpness’’ of the apex. A small λ value produces a sharp apex while a large value produces a rounded apex. Readers are referred to Wang (1982) for details.

By rotating this conical curve about the z axis we obtain a conical body of revolution whose horizontal (x–y) cross section is a circle of radius a. This expression has been used by Wang et al. (1987) to characterize the size and shape of an ensemble of hailstones. While many hailstones have approximately circular cross sections, many have obvious elliptical cross sections (C. A. Knight 1995, personal communication; A. Waldvogel 1995, personal communication). In order to represent those stones with elliptical cross sections, we can modify the three-dimensional version of Eq. (9) to arrive at the following expression:

$$\frac{x^2}{[a\cos^{-1}(z/\lambda c)]^2} + \frac{y^2}{[b\cos^{-1}(z/\lambda c)]^2} + \frac{z^2}{c^2} = 1. \quad (10)$$

The horizontal cross section of this conical particle at a vertical level $z = z_o$ is an ellipse whose semiaxes in the x and y directions are $a\cos^{-1}(z_o/\lambda c)$ and $b\cos^{-1}(z_o/\lambda c)$, respectively. Figure 6 shows an example of such a particle.

4. Potential applications

The shape-describing equations given above have certain obvious applications. One is using them to characterize an ensemble of ice particles. These equations involve only a few characteristic parameters and it is relatively easy to determine their values given a certain sample of these particles. This can be done manually or by an automated process. Wang et al. (1987) employed an automated process to analyze several hundred hailstones based on Eq. (9). They demonstrated that the distributions of parameters a, c, and λ can indeed be used to characterize the shape and size of nonspherical particles. Equation (10) only involves one more parameter, b, than Eq. (9) and, hence, such an automated process can be easily designed and employed. Obviously, Eqs. (6) and (8) can be utilized for the same purpose.

These equations can also be used to define the boundary conditions for solving various theoretical problems involving nonspherical ice particles in the atmosphere [e.g., the scattering cross section of conical particles as done by Sturniolo et al. (1995) but it was done for conical bodies of revolution]. For instance, due to the difficulty of describing the shape of these particles by analytical equations, the boundary conditions for light scattering by ice crystals are usually specified in a ‘‘hard’’ numerical way. The shape of an ice crystal is drawn on paper, and the coordinates of the points on the surface are determined manually. With equations given in this paper, it should be much easier to determine these coordinates and, hence, the boundary conditions. In addition, it is also possible to design numerical grids based on these equations for numerically solving light scattering, Navier–Stokes, and diffusion equations involving these nonspherical ice particles.

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