NOTES AND CORRESPONDENCE

Characterization of Ice Crystals in Clouds by Simple Mathematical Expressions Based on Successive Modification of Simple Shapes

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ABSTRACT

The technique based on the concept of successive modification of simple shapes using elementary mathematical functions to represent the shape and size of ice crystals in clouds is discussed. Two hypothetical samples of ice crystals, a single-habit sample of hexagonal plates and a multihabit crystal sample, are generated using a formula developed previously to illustrate the use of this technique in generating ice crystal ensembles in cloud models. Next, a new expression representing columnar ice crystals is described. Finally, two new expressions that can be used to generate the three-dimensional combination of bullets and spatial dendrites are described. The parameters involved in these expressions are expected to be useful in characterizing the shape and size spectra of ice crystals found in cirrus clouds.

1. Introduction

In a series of papers, Wang (1982), Wang and Denzer (1983), and Wang (1987) developed simple mathematical formulas that can be used to describe both the shape and size of nonspherical cloud and precipitation particles. These formulas can be utilized for various purposes. For example, Heymsfield (1972) determined the bulk densities of dendritic ice crystals by mean of the ratio of particle area to the area of an equivalent circle times the density of bulk ice (0.91 g cm^{-3}). This required the use of a planimeter to estimate the crystal area, which could be rather tedious work when the number of crystals to be measured is large. By representing the shape and size of these particles by simple mathematical functions, the area can be easily determined by integrating this function (Wang and Denzer 1983), and it is possible to develop computerized techniques for such routine tasks. Another usage is the specification of quantitative boundary conditions, which are necessary for solving theoretical problems involving differential equations such as the scattering of sunlight by ice crystals in cirrus clouds (e.g., Liou 1993).

The present study is an extension of these earlier works. In the following section, hypothetical samples of ice crystals will be generated using these expressions to illustrate the use of this technique to characterize an ensemble of particles. Secondly, a mathematical expression for describing the size and shape of columnar ice crystals, which was not discussed previously, will be presented. Finally, extension of this technique to describe the shape and size of three-dimensional ice particles will be discussed.

2. Two-dimensional characterization of planar hexagonal ice crystals

The method used in this and previous studies can be termed as the successive modification of simple shapes (SMOSS). It is based on the concept that complicated shapes can be generated from simple ones by successively applying modifying mathematical functions. For example, all planar ice crystals have the basic shape of a plate that, in its simplest form, is a circular disk. In two-dimensional form, the circular disk becomes a circle. By successive modification by some sinusoidal functions, the circle can be ‘‘smossed’’ into more complicated shapes. An expression based on this technique as given by Wang and Denzer (1983) and Wang (1987) is, in polar coordinates,

$$r = a[\sin^2(n\theta)]^b + c, \tag{1}$$

where r and θ are the radial and angular coordinates, respectively; a, b, c, and n are adjustable parameters to fit the shape and size of the ice crystals. The ranges of these parameters are

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a: from $-c$ to ∞ (amplitude parameter)
b: from 0 to ∞ (width parameter)
c: from 0 to ∞ (center size parameter)
n: 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\cdots$ (polygonality parameter).

For the characterization of ice crystals, $n = 3$ since the majority of ice crystals are hexagonal. Examples of ice crystal habits generated by (1) can be found in the papers mentioned above. The method of fitting observed crystal samples has also been discussed in these papers and hence will not be repeated here.

In addition to the use of characterizing actual ice crystal or snow samples, (1) can be used to generated "model" samples of ice crystals in clouds. The latter may be particularly useful for cloud modeling work. Recent cloud models often contain detailed microphysics for describing the growth of or interactions between various kinds of cloud particles (see, e.g., Cotton and Anthes 1989; Johnson et al. 1993, 1994). Up to now most models only implement generic categories of ice particles without specifying their habits. Yet crystal habits do influence the diffusional and collisional growth rates (see, e.g., Pruppacher and Klett 1978; Wang and Ji 1992), and also the radiative properties of clouds. To include crystal habit feature in the model quantitatively, one can use the mathematical formulas mentioned here. First, the model would determine the temperature and saturation ratio of a particular cloud region. Then an appropriate crystal habit can be selected according to the temperature and saturation ratio (see p.30 of Pruppacher and Klett 1978). Suppose only one kind of ice habit, say, hexagonal plate, is to be assigned in a certain cloud region, then the parameter b in Eq. (1) is fixed. Next, the amount of excess moisture that is to be converted into ice is distributed into specified spectra of a and c. The distributions of a, b, and c thus specified determine not only the size but also the shape of ice crystals in this region.

Figure 1 shows such a hypothetical ensemble of hexagonal plates whose distributions of a, b, and c are given in Fig. 2. Since these are congruent shapes, b is a single peak, that is, $b = 0.397$ (see Wang 1987). In this example, a and c distributions are chosen to resemble gamma-type distributions. Note that in this sample the value of a/c is fixed.

It is also possible to use this technique to generate a sample of ice crystals with many different habits. Figure 3 shows another hypothetical sample of ice crystals generated by (1) whose distributions of parameters a, b, and c are shown in Fig. 4. The choice of the spectra is arbitrary, just to illustrate the usage of this technique. Due to the wider spectra of these parameters (especially b), there are more varieties of crystal habits generated.

At present it is not known what the typical spectra of a, b, and c should look like in a particular region of a cloud. It obviously requires a large amount of observations to determine them. It will be necessary to develop automatic (computerized) techniques to analyze the actual samples or the two-dimensional images obtained by cloud particle probes such as the Particle Measuring Systems (PMS) 2DC or 2DP probes (see, e.g., McFarquhar and Heymsfield 1996).

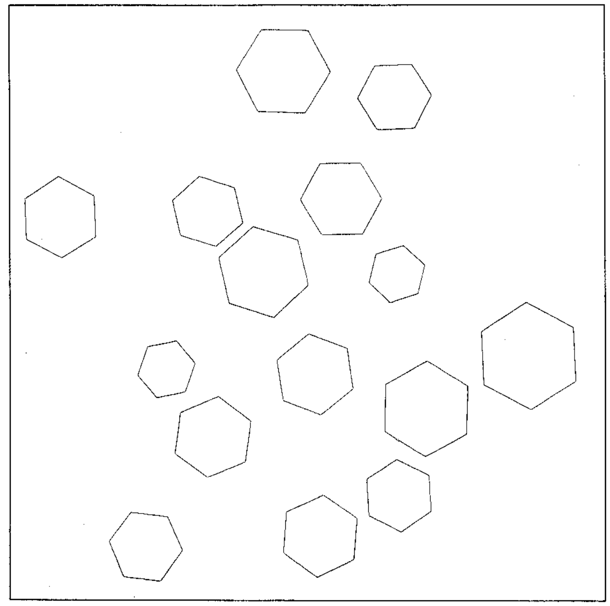


FIG. 1. A simulated sample of 15 hexagonal ice plates generated by Eq. (1).

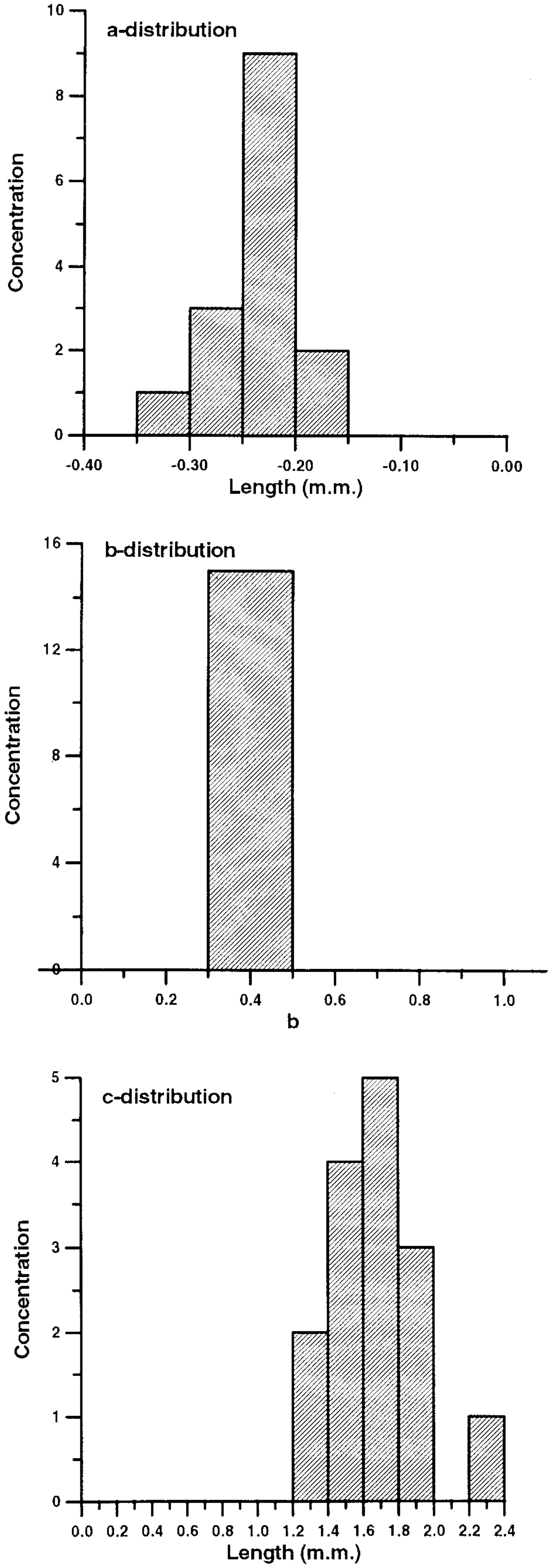
3. Mathematical description of columnar crystals

Another common habit of ice particles in cirrus clouds is the columnar crystal. It is, in fact, fairly easy to characterize a columnar crystal (assuming, of course, that the crystal is a simple column) by simply determining its length (L) and diameter (D). But we can also use the SMOSS technique to generate the shape of a columnar crystal. We note that the two-dimensional projection of a columnar crystal (approximated by a circular cylinder of finite length) is a rectangle that can be obtained by smossing an ellipse whose semi-axes are ($L/2$) and ($D/2$), respectively. Assuming the length of the column lies along the x axis, then the following formula is the required expression:

$$\frac{4x^2}{L^2} + \frac{4y^2}{D^2}\left(1 + \varepsilon - \frac{4x^2}{L^2}\right) = 1, \qquad (2)$$

where ε is very small positive number ($\varepsilon \ll 1$). When $\varepsilon = 0$, (2) represents the exact equation for the rectangle everywhere except at $x = \pm L/2$ where discontinuities occur. With a small positive ε, (2) generates an approximated rectangle with length L and height D without the problem of such discontinuity. Figure 5 shows a rectangle generated by (2) for $L = 4$, $D = 2$, and $\varepsilon = 10^{-5}$.

Note that ε can be set as small as one wishes. The smaller the ε, the closer Eq. (2) can represent a real



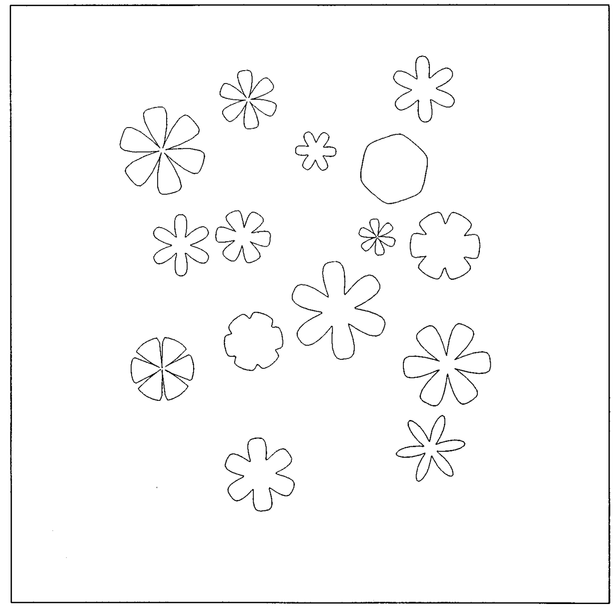


FIG. 3. A simulated sample of 15 ice crystals generated by Eq. (1) with the distributions of *a*, *b*, and *c* as given in Fig. 4.

rectangle. Note also that a three-dimensional circular cylindrical column can be generated by rotating the rectangle about the x axis.

4. Three-dimensional characterization of ice crystals in clouds

Equation (1) describes only the two-dimensional shapes of ice crystals. This may be adequate for simple planar crystals (although the thickness parameter is neglected) but obviously falls short in describing many ice crystals whose characteristics are conspicuously three-dimensional, for example, C2a, P5b, P7b, and CP2a in the Magono–Lee classification (Magono and Lee 1966). Some of these crystals may have relatively high frequency of occurrence, for example, bullets in cirrus clouds. Before we try to devise formulas to describe the three-dimensional ice crystals, it is useful to briefly review their structures.

Three-dimensional ice crystals are thought to grow from frozen drops and hence are usually polycrystalline. They are not well understood relative to single crystals and only a handful of studies exist (see, e.g., Hallett 1964; Lee 1972; Hobbs 1976; Kobayashi et al. 1976a; Kikuchi and Uyeda 1979; Furukawa 1982). Unlike single crystals treated in previous sections, these crystals are usually not polygonally symmetric. Even with a single type of crystals, for example, the combination of

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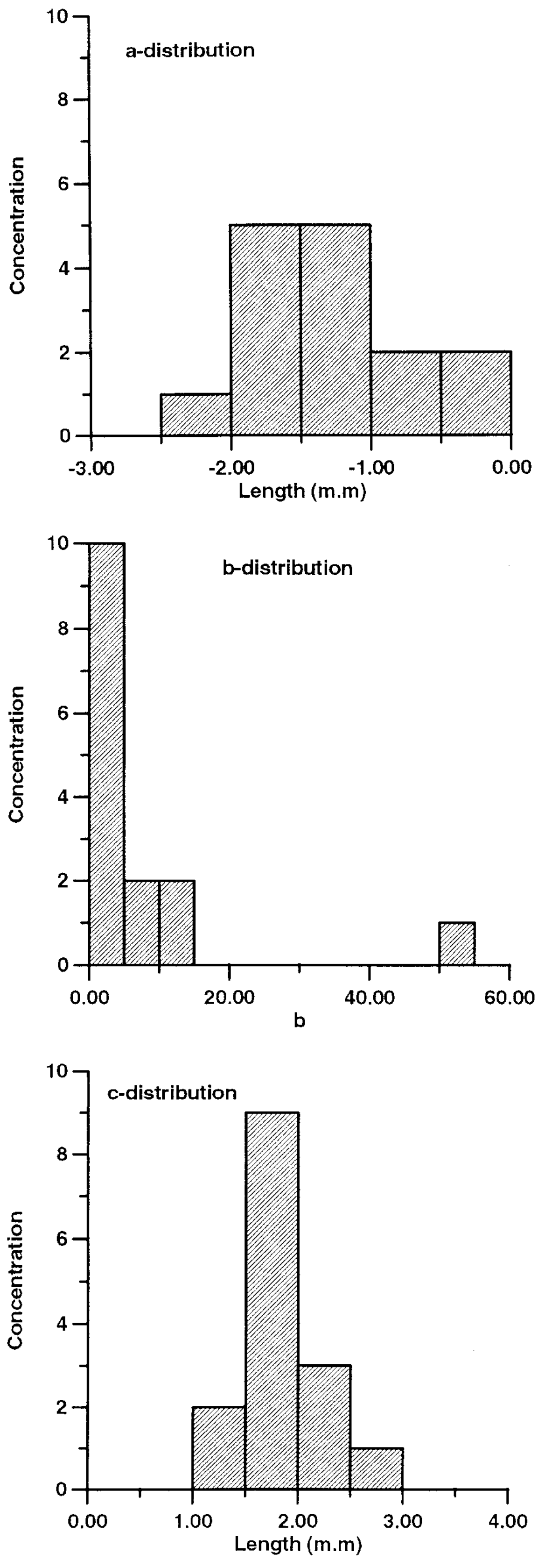
FIG. 2. Distributions of *a*, *b*, and *c* for the simulated sample shown in Fig. 1. In this example b = 0.397.



bullets, the angle between different branches may differ widely in different samples. Figure 6 is an example of the frequency histograms of angles between the *c* axes of each component for the case of bullets (from Kobayashi et al. 1976a). Strangely, the other two more common types of three-dimensional crystals, namely, the spatial dendrites and radiating dendrites, both have dominant angle frequency at 70° (Lee 1972; Kobayashi et al. 1976a,b; Kikuchi and Uyeda 1979).

While it may be possible to simulate rather closely the shape of these three-dimensional crystals, the ensuing mathematical expressions would be rather complicated, especially in reproducing the 70° angle, which cannot be obtained by evenly dividing 360°. Since the purpose of introducing mathematical formulas here is to use them for estimating the bulk distributions of ice water contents and performing first-order calculation of some simple diffusional and radiative properties, not for precise crystallographic investigations, it is felt that simple formulas that produce polygonally symmetric shapes instead of shapes with 70° angles are probably sufficient. In addition, at this moment it seems that the collection, preservation, and analysis of a large sample of three-dimensional crystals would be a rather intractable effort, so it is probably impractical trying to characterize an actual sample. Perhaps in the future the holographic technique will advance far enough so that such tasks can be easily and economically done. On the other hand, it is relatively easy to construct hypothetical samples with characteristics similar to actual samples. It is for this purpose that the formulas to be described below are designed.

With the above considerations in mind, it is possible to generalize the two-dimensional equation (1) to three dimensions so as to simulate two more commonly observed three-dimensional ice crystals, namely, combination of bullets and radiating dendrites. These expressions are described in the following.

Bullets are fairly common in cirrus clouds (see, e.g., Heymsfield 1975; Parungo 1995). Cirrus clouds are known to have considerable influence on the radiative budget and hence the climate of the earth–atmosphere system; therefore it is of significance to examine the expressions that can simulate such a shape. The following expression can be used to generate the combination of bullets:

$$r = \{a[\cos^2(m\theta)]^b + c\}^d\{a'[\sin^2(n\varphi)]^{b'} + c'\}^{d'}. \quad (3)$$

This equation is simply Eq. (1) applied to both the θ and φ directions. Thus, it can be expected that it will produce shapes similar to those shown in previous sections when looking at some specific θ or φ cross sections. The shape generated by this expression will have

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FIG. 4. Distributions of *a*, *b*, and *c* for the simulated sample shown in Fig. 3.

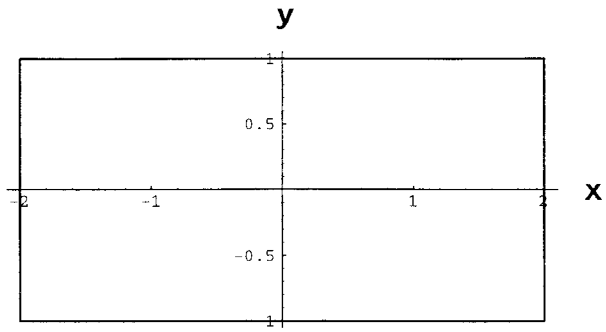

FIG. 5. A 2D columnar crystal generated by Eq. (2) with $L = 4$ and $D = 2$.

$2mn$ branches. For example, a four-branch combination of bullets as shown in Fig. 7a can be generated by the expression

$$r = [1 - \cos(2\theta)^4]^{20}[1 - \sin(\varphi)^4]^{20}, \qquad (4)$$

where $m = 2$ and $n = 1$. The angle between branches is evidently 90°, which also occurs in nature, as indicated by Fig. 6. The width of the branch is controlled by b and b' in Eq. (3). For example, if we change the values of $b = b' = 4$ in Eq. (3) to $b = b' = 6$, then the branches will look "fatter" as shown in Fig. 7b. In both examples, the branches have relatively flat end surfaces that are close to the actual samples. However, there are cases where the end surfaces are capped plates that are not simulated here. The shape of each branch is not hexagonal as would be the case for real ice bullets. It is unclear at this point whether this really matters much in terms of the bulk radiative properties.

Figure 7c is an example of a bullet combination with 8 ($m = n = 2$) branches. The angle between branches is again 90°, but this time the branches are distributed in two mutually perpendicular planes. A "broken" combination, say, only the lower half, can be generated by selecting only the values of θ from $\pi/2$ to π.

The other habit to be considered here is the radiating dendrites. Again, it is impractical to simulate the intricate designs of each branch, but the basic shape of the crystal can be represented by the expression

$$r = a[\sin^2(m\theta)]^b[\sin^2(n\varphi)]^{b'} + c. \qquad (5)$$

This is an obvious extension of Eq. (1) from 2D to 3D

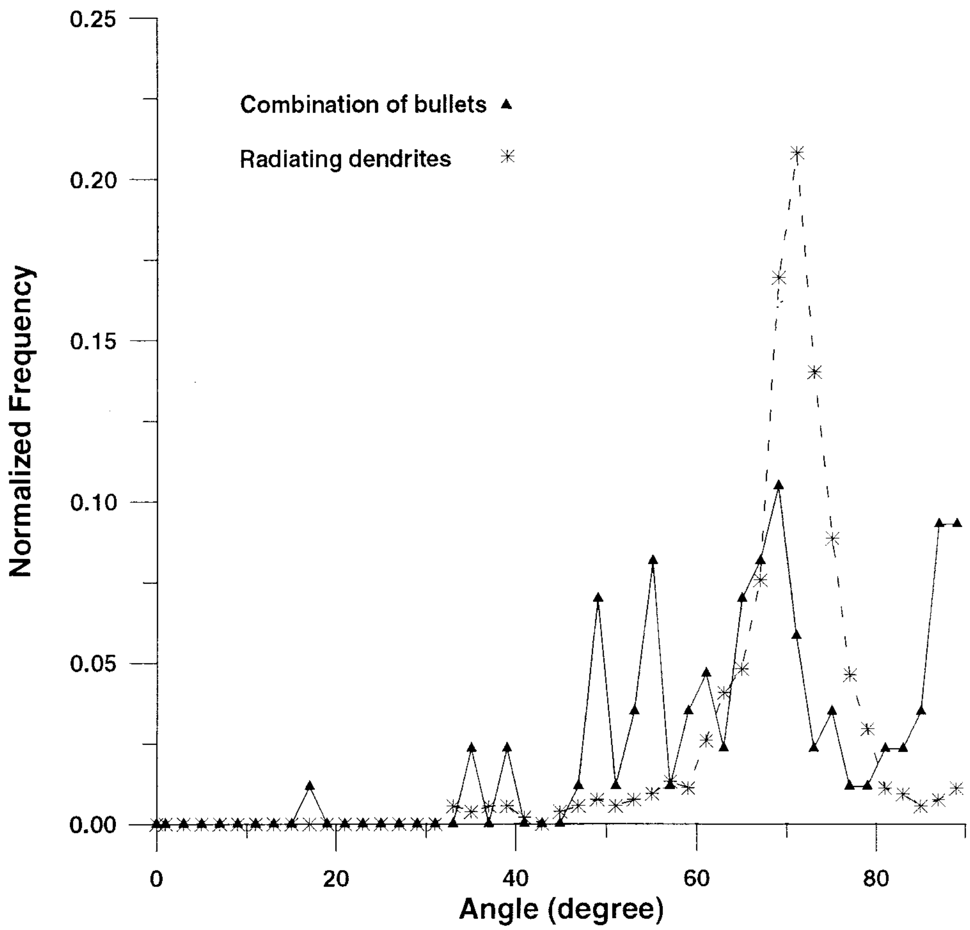

FIG. 6. Normalized frequency of occurrence of the angle between two branches of combination of bullets and radiating dendrites. [Data adopted from Kobayashi et al. (1976a).]

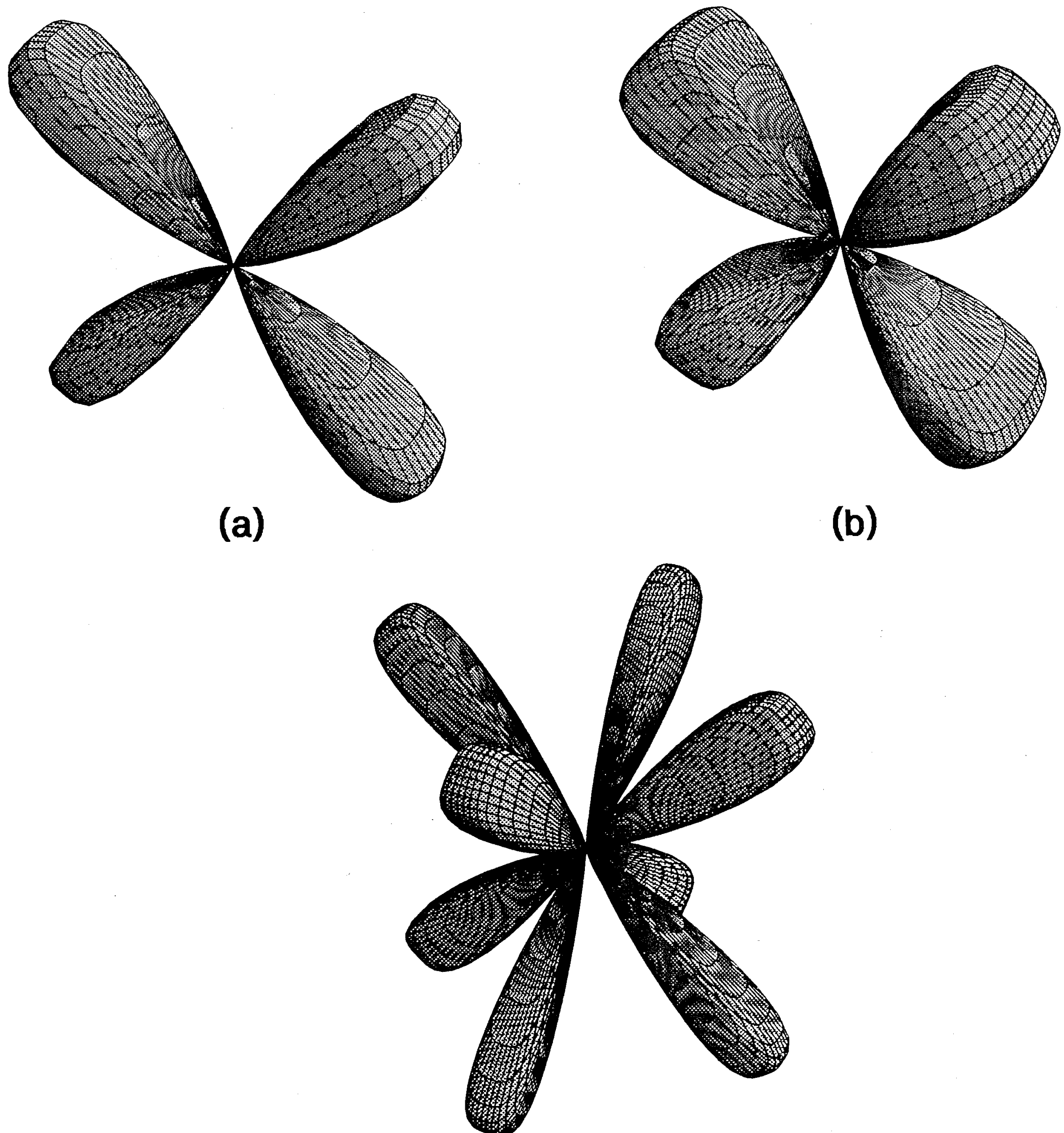

FIG. 7. (a) A four-lobe combination of bullets generated by Eq. (3). (b) Another example of four-lobe combination of bullets generated by Eq. (3) except $b = b' = 6$ in this case. (c) An example of an eight-lobe combination of bullets generated by $r = [1 - \cos(2\varphi)^4]^{20}[1-\sin(2\theta)^4]^{20}$.

and hence it is easy to see that it will produce 3D polygonally symmetric shapes. Figure 8 is an example of radiating dendrites generated by Eq. (5) by assigning $a = 0.1$, $b = b' = 30$, and $c = 0.001$. The large values of b and b' are to make the branches very thin.

Fitting observed 3D crystals by the above expressions is analogous to that for 2D crystals except the measurements may be difficult to perform as mentioned previously. The easiest way to do this is probably measuring the 2D projection of the crystals and then proceeding to do the fitting process for the 2D crystals as described in Wang and Denzer (1983) and Wang (1987). Since the expressions given in this paper generate polygonally symmetric shapes, it should be easy to obtain a 2D projection.

5. Summary

It is seen in previous sections that by applying the SMOSS technique, we can generate certain shapes that simulate the basic shapes of both two- and three-dimensional ice crystals observed in clouds. For two-dimensional crystals, it is relatively easy to use these formulas to fit actual samples so that the geometrical char-

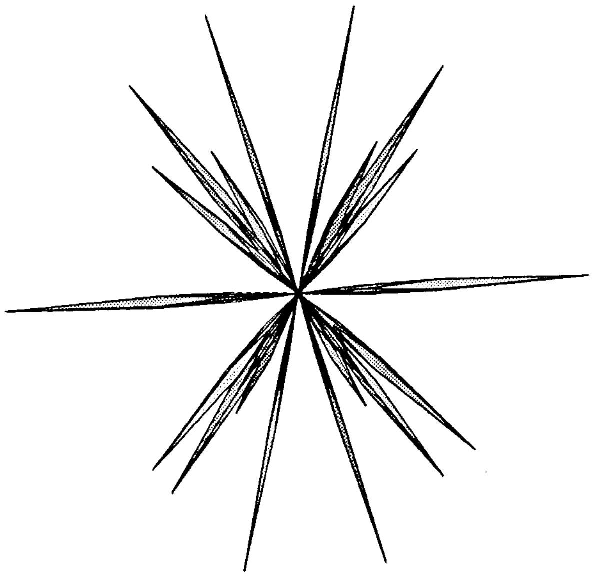

FIG. 8. A radiating dendrite generated by Eq. (4) with $a = 0.1$, $b' = 30$, and $c = 0.001$.

acteristics of these samples can be expressed in terms of the frequency distributions of the adjustable parameters. Conversely, the formulas can be used to generate typical ensembles of ice crystals in a cloud model. For three-dimensional crystals, the fitting of actual samples is more difficult but the generation of typical ensembles is still relatively simple. The next logical task is to produce typical ensembles of both two- and three-dimensional ice crystals with characteristics of actual samples. This is currently being studied and hopefully the results will be reported in the future.

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