The capacitance of solid and hollow hexagonal ice columns

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[1] The capacitances of solid and hollow hexagonal ice columns are calculated using the classical electrostatic analogy theory. Finite element techniques are used to solve the Laplace equation to obtain water vapor density distribution from which the capacitance can be determined. The results show that solid and hollow columns of the same dimensions have nearly the same capacitance despite the existence of cavities in the latter, which implies the same mass growth rates of the two. The computed capacitances agree well with experimental measurements. Since the volume of a hollow column is smaller than that of a solid column of the same dimensions, the same mass growth rate prompts the hollow column to grow faster in linear dimensions and hence interact stronger with radiation. This will have important implications on the cirrus influence on climate. **Citation:** Chiruta, M., and P. K. Wang (2005), The capacitance of solid and hollow hexagonal ice columns, *Geophys. Res. Lett.*, *32*, L05803, doi:10.1029/2004GL021771.

1. Introduction

[2] Hexagonal columnar ice crystals are one of the most widely distributed ice crystal habits in atmospheric clouds [*Pruppacher and Klett*, 1997; *Wang*, 2002; *Walden et al.*, 2003]. *Heymsfield and Platt* [1984] and *Heymsfield and Iaquinta* [2000] reported that the ice crystals sampled in high cirrus clouds (with cloud temperature $<-50°C$) are largely such columns. Cirrus clouds are thought to have great impacts on global climate due to their strong interaction with solar and terrestrial radiations [*Ramanathan et al.*, 1983; *Liou*, 2002].

[3] *Liu et al.* [2003a, 2003b] studied the effects of ice crystal microphysical properties on the development of cirrus clouds using a 2-D cirrus dynamic model and found that the heating rates of cirrus by both solar and terrestrial radiations are sensitive to ice habit. They found that, aside from its direct effect on the scattering of radiation, ice habit also influences the cloud radiative properties via its control on the ice growth rate. Ice crystals of different habits grow at different rates in general. In an environment of fixed water vapor supply, the ice growth rate determines the crystal sizes and concentrations in the cloud, which, in turn, determine the cloud radiative properties. Thus, even if the initial and boundary conditions are identical, a cloud that develops into a cirrus consisting of columns may possess different radiative properties than one that develops into a cirrus of bullet rosettes not only due to the different scattering cross-sections but also the different growth rates of their constituent ice crystals.

[4] In-situ sampling of ice crystals in cirrus clouds show that about 90% of the columnar crystals are hollow (K. N. Liou, personal communication, 2004). They differ from solid columns in that they have cavities in the basal planes. While the size, shape and orientation of the cavities vary from column to column, a typical case is that the cavities are present in a symmetric manner, namely, one cavity on each side along the length (c-axis) of the column (see Figure 1). The cross-section of the cavity viewed from the side is approximately triangular.

[5] Because of the cavities, hollow columns may grow at different rates than solid columns. They also have different optical properties [*Liou*, 2002]. Given the wide presence of hollow ice columns in cirrus clouds, it is perhaps surprising that their growth rates have never been determined either experimentally or theoretically. The need to determine these growth rates motivated this study, which will focus on the theoretical methods.

[6] The traditional technique of theoretically determining the growth rates of ice crystals is the electrostatic analogy [see, e.g., *McDonald*, 1963; *Pruppacher and Klett*, 1997]. The central quantity in question in this technique is the capacitance of the ice crystal. Surprisingly, even the capacitances of relatively simple ice habits, such as solid hexagonal columns and plates, have never been theoretically determined in a rigorous manner but only approximately. This is mainly due to the difficulty in describing the shape of hexagonal columns and plates by simple continuous mathematical functions because of their sharp edges. Consequently, solid hexagonal ice columns and plates were approximated by prolate spheroids and thin oblate spheroids respectively, and used the capacitances of these approximated crystals to represent real ice columns and plates [see *Pruppacher and Klett*, 1997]. In the present study, the capacitances of both solid and hollow hexagonal columns will be calculated using the exact shapes.

2. Mathematical Formulation

[7] The technique of determining the capacitance of ice crystals used in the present study is the same as that employed by *Chiruta and Wang* [2003] for determining the capacitance of bullet rosette ice crystals, hence we will only give a brief sketch of the mathematical procedure below. The ice crystals are assumed to be stationary and no ventilation effect will be considered.

[8] We first determine the distribution of water vapor density ρ_v around a stationary ice crystal of arbitrary shape. Under this condition, the vapor density satisfies the Laplace equation:

$$\nabla^2 \rho_v = 0 \tag{1}$$

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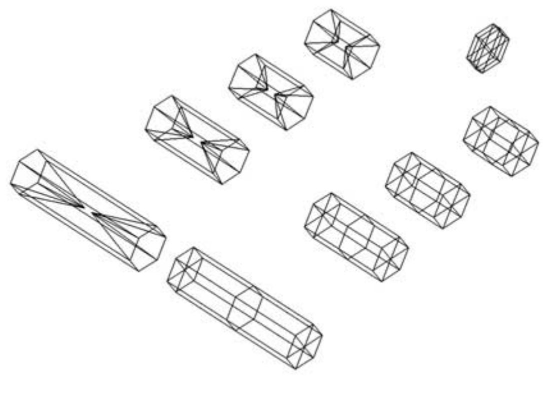

Figure 1. The nine simulated columnar ice crystals.

and the following boundary conditions:

$$\rho_v = \begin{cases} \rho_s \text{ at the crystal's surface} \\ \rho_\infty \text{ far away from the crystal} \end{cases} \quad (2)$$

where both ρ_s and ρ_∞ are constant. ρ_s is the same as the saturation vapor pressure.

[9] Once the vapor density distribution ρ_v is determined, the capacitance can be obtained by following the same procedure as given by *Chiruta and Wang* [2003]. *Smythe* [1956, 1962] and *Wang et al.* [1985] performed similar calculations to determine the capacitances of right circular cylinders of finite lengths by solving the Laplace equation analytically. The present study uses numerical techniques instead due to the more complicated shapes.

3. The Columnar Crystals and Numerical Discretization

[10] Figure 1 shows the nine hexagonal columnar ice crystals investigated in the present study. The eight crystals are divided into four solid and hollow pairs; the crystals in each pair have the same dimension and aspect ratio R (= c/a where a is the radius of the column on the basal plane and c is the column's half-length).

[11] The hollow columns studied here are idealized from actual ice column photographs. Surface irregularities are ignored and the cavities are assumed to taper towards the center to form inward-pointing cones, resembling an hourglass shape. The cavities do not reach the center of the column. The small hexagonal disk in Figure 1 is a disk whose thickness is the distance between the tips of the two opposing cavities. Its capacitance is also calculated to serve as a reference. The aspect ratios of these crystals are (from right to left): R = 0.2, 1.1, 1.36, 1.9, and 3.33.

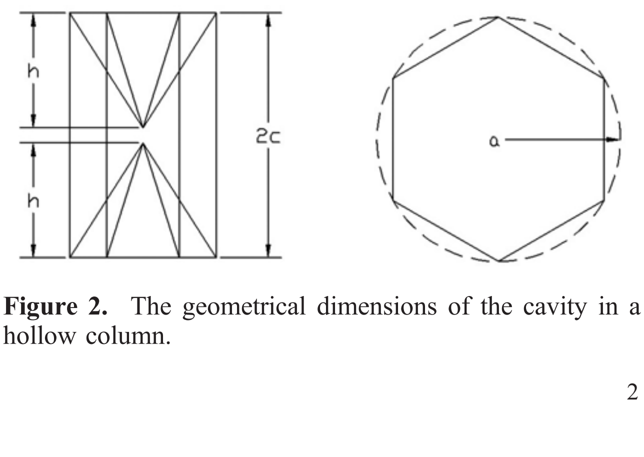

Figure 2. The geometrical dimensions of the cavity in a hollow column.

[12] The outer rims of the cavities are assumed infinitely sharp. Detailed geometrical specifications of the columns and the cavities are given in Figure 2. As shown in Figures 1 and 2, the cavities look like six-sided pyramids with hexagonal bases.

[13] The surface area and volume of these crystals can be calculated using the following formulas:

Solid columns

$$S = 12ac + 3\sqrt{3}a^2 \quad (3)$$

$$V = 3\sqrt{3}a^2c \quad (4)$$

Hollow columns

$$S = 12ac + 6a\sqrt{h^2 + \frac{3}{4}a^2} \quad (5)$$

$$V = 3\sqrt{3}a^2c - \sqrt{3}a^2h \quad (6)$$

where S and V are the surface area and volume, and h is the depth of the cavity (see Figure 2).

[14] Finite element techniques similar those employed by *Chiruta and Wang* [2003] are employed to numerically solve the Laplace equation (1). Figure 3 shows an example of the hollow column mesh used for the numerical calculations.

4. Results and Discussions

4.1. Capacitance Results

[15] The capacitances of four solid columns, four hollow hexagonal ice columns and the reference disk calculated using the method described above are plotted as a function of the aspect ratio R as shown in Figure 4. Experimental measurements of solid and hollow metal hexagonal columns (R = 2.63) by *Podzimek* [1966] are also shown, which agree excellently with our calculations. Also plotted on the chart are capacitances of prolate/oblate spheroids (from the for-

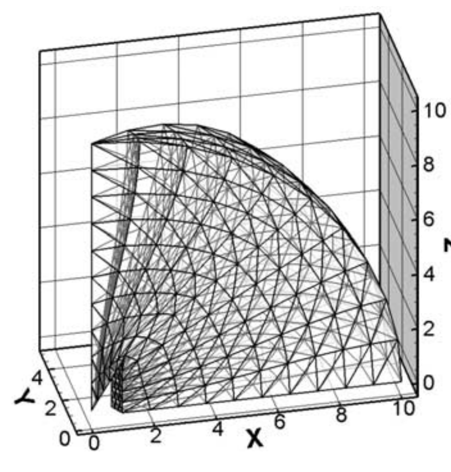

Figure 3. The discretization into finite elements of the analysis domain for a hollow column. Due to the symmetry, only 1/12 of the column (the portion close to the origin) is needed to form the mesh.

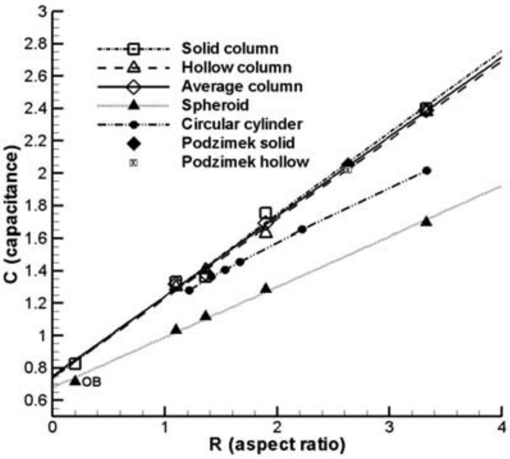

Figure 4. The capacitances of hexagonal ice columns, prolate and oblate (the one indicated by OB) spheroids, and circular cylinders as a function of the aspect ratio R. Experimental results are taken from *Podzimek* [1966].

mulas given by *Pruppacher and Klett* [1997, pp. 547–548]) and calculated circular cylinders [from *Wang et al.*, 1985] for comparison. First, the capacitance appears to be a linear function of the aspect ratio for both solid and hollow columns. Thus for a fixed radius a, the longer the column, the greater is the capacitance.

[16] Secondly, the differences between the capacitance values of solid and hollow columns of the same aspect ratio are very small, and are either due to the slight difference in geometry or grid resolution. In essence, the differences are too small to be significant. Thus, we propose to use the average values as representing the capacitance for both solid and hollow hexagonal columns. The average capacitances as a function of the aspect ratio can be fitted into a line as shown in Figure 4 and the fitting equation is:

$$C = 0.751 + 0.491R \tag{7}$$

where C is the capacitance. All the following discussions about column capacitances refer to values calculated using (7).

[17] Figure 4 also shows that using the capacitances of circular cylinders and prolate spheroids to approximate that of hexagonal columns of the same aspect ratios result in errors of ~20% to ~30% respectively for $R > 3$ and seem to increase with increasing R. The errors for smaller aspect ratios are smaller.

[18] Figure 5 shows the capacitance of solid and hollow hexagonal ice columns as a function of surface area and volume. The data points in Figure 5 can be fitted by the following linear equations:

$$C = 5.672 \times 10^{-1} + 4.133 \times 10^{-2} S \quad \text{(Solid)} \tag{8}$$

$$C = 6.790 \times 10^{-1} + 2.897 \times 10^{-2} S \quad \text{(Hollow)} \tag{9}$$

$$C = 7.016 \times 10^{-1} + 1.418 \times 10^{-1} V \quad \text{(Solid)} \tag{10}$$

$$C = 7.057 \times 10^{-1} + 9.451 \times 10^{-2} V \quad \text{(Hollow)} \tag{11}$$

where S are in a^2 unit and V in a^3 unit, respectively.

[19] From hindsight, it is perhaps reasonable to expect the similar-dimensioned hollow and solid columns to have similar capacitances. In the electrostatic analogy theory, the ice crystals are treated as perfect electric conductors. If the hollow column's base is closed by a very thin surface instead of open as in the present case, we would expect its capacitance to be the same as that of a solid column of the same dimension and aspect ratio because all electric charges would exist only on the outer surface irrespective of how large the cavity is inside. Now if the thin surface has some openings so that the surface is no longer closed, we would probably expect that the capacitance should still remain nearly unchanged because the surface is very thin and nothing substantial is altered in the new configuration. Currently we do not have a rigorous proof of this conjecture, but the numerical results seem to lend support to it.

4.2. Implications of the Capacitance Results

[20] Because of the cavities, a hollow column has a greater surface area and a smaller volume than a solid column of the same dimension and aspect ratio. Figure 6 illustrates this statement clearly.

[21] It is seen here that, as the aspect ratio increases from 1 to 4 (i.e., the length of the column increases from c to $4c$), the solid-to-hollow surface area ratio drops from ~0.84 to ~0.70 whereas the corresponding volume ratio increases from ~1.37 to ~1.48. The difference is the greater the longer the columns. The equations for the linear fits in Figure 6 are:

$$(S_s/S_h) = 8.857 \times 10^{-1} - 4.469 \times 10^{-2} R \tag{12}$$

$$(V_s/V_h) = 1.352 + 3.204 \times 10^{-2} R \tag{13}$$

where S_s and S_h are the surface area, and V_s and V_h are the volume of solid and hollow columns respectively.

[22] The difference in volume between the solid and hollow columns is especially relevant to the discussion of growth rate. According to the electrostatic analogy, a hollow column will have the same mass growth rate (dm/dt) as a solid column having the same capacitance C. If the two

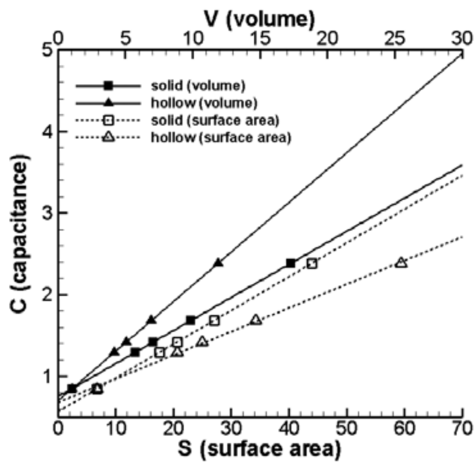

Figure 5. Variation of capacitance with column surface area and volume.

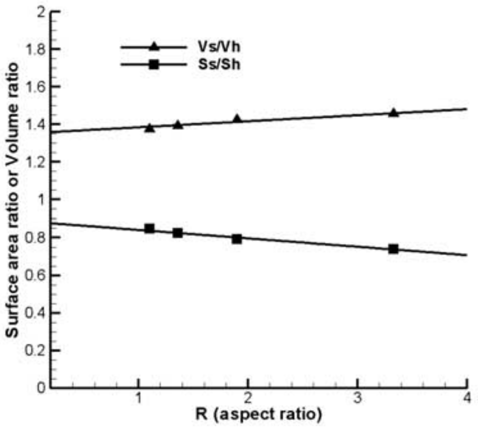

Figure 6. Hollow to solid surface area ratio (S_h/S_s) and volume ratio (V_h/V_s) as a function of aspect ratio.

have the same dimension and aspect ratio, then the hollow column has a smaller volume than the solid column. If we assume that the ice densities of the two ice columns remain the same, then since $dm/dt = \rho_{ice}(dV/dt)$ where ρ_{ice} is the ice density, the volume growth rates of the two columns are also the same. But the same dm/dt implies that the hollow column will grow faster in linear dimension—either it grows longer or thicker or both, and that will eventually lead to a different dimension and aspect ratio from the solid column. From then on, of course, even the mass growth rates will be different since the capacitance will be different. If vapor supply remains steady, the capacitance of the hollow columns will become greater than that of the solid columns and hence grow at higher mass rates. The same capacitance also determines the evaporation rate, just that the vapor flux is in the opposite direction. Equations (7)–(11) provide the capacitance values of solid and hollow hexagonal ice columns in various forms that can be used in cirrus cloud models such as *Liu et al.* [2003a, 2003b].

5. Conclusions and Outlooks

[23] We showed above that the capacitances of solid and hollow hexagonal ice columns of the same dimension and aspect ratio are practical the same. We have developed empirical formulas of the crystal capacitance as a function of aspect ratio, surface area and volume of the ice crystal. These formulas can be employed to determine the columnar ice crystal growth and evaporation rates.

[24] Of course, the capacitance is not the only factor determining the ice growth rates. The ice growth and evaporation rates in clouds depend on other factors in addition to the capacitance. During the growth or evaporation of an ice crystal, latent heats will be released or consumed because phase change of water substance is occurring. Thus the temperature at the crystal surface will be warmer or colder than its environment depending whether deposition or evaporation is occurring, and this temperature influences the saturation vapor density value and hence the growth/evaporation rate. Thus, the determination of growth rate is a coupled heat and mass transfer problem [*Pruppacher and Klett*, 1997]. Another factor is the ventilation effect caused by the motion of the crystal in air [see, e.g., *Pruppacher and Klett*, 1997; *Ji and Wang*, 1998]. We are currently performing calculations of the growth and evaporation rates and bulk densities of both solid and hollow columnar ice crystals with the above two factors included and the results will be reported in the near future.

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